Semiautomatic Ordinal and Ring Structures

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Slides

https://m4th.b0ss.net/semi/semi.pdf

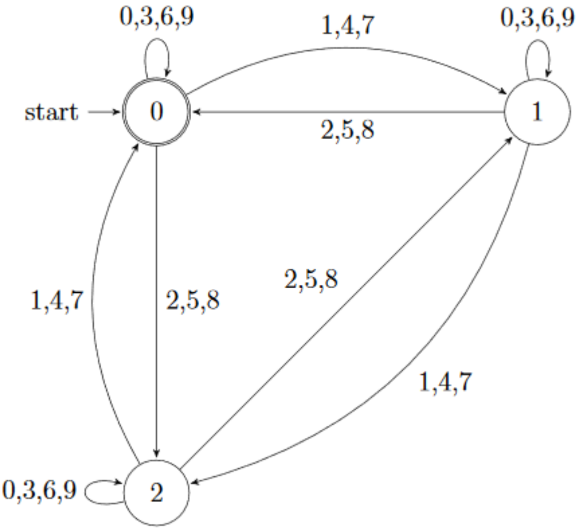
Background

Automata theory

Finite automata

- Processes input symbol by symbol with finite memory
- The finite memory indicates whether the word as seen so far satisfies the condition to be checked.

Check multiple of 3



Automatic relations

- Encode idea of process tuples in parallel
- Add a padding symbol $\#$
- $conv(010, 01236) = \binom{0}{0}\binom{1}{1}\binom{0}{2}\binom{\#}{3}\binom{\#}{6}$
- A function is automatic iff its graph (encoded this way) is automatic

Verifying addition

Primary school algorithm

- n – correct and no carry
- c – carry
- i – wrong

```
Correct addition     Incorrect addition
      2 3 5 8              3 3 3 3
      # 1 1 2              # # 2 2
      2 4 7 0              # 1 5 5
     n n n c n            i i n n n
```

Automatic structures

A structure $(A, f_1, \dots, f_n, R_1, \dots, R_m)$ is **automatic** iff

- A is a regular set,
- $f_1, \dots, f_n$ are automatic functions,
- $R_1, \dots, R_m$ are automatic relations.

Example

$(\mathbb{N}, +)$ is automatic but $(\mathbb{N}, +, \cdot)$ is not automatic.

Semiautomatic structures

$(A, f_1, \ldots, f_n, R_1, \ldots, R_m; g_1, \ldots, g_p, S_1, \ldots, S_q)$ is **semiautomatic** iff

- A is a regular set,
- $f_1, \ldots, f_n$ are automatic functions and $R_1, \ldots, R_m$ are automatic relations,
- $g_1, \ldots, g_n$ are semiautomatic functions and $S_1, \ldots, S_m$ are semiautomatic relations,

where $f : A^n \to A$ is semiautomatic iff fixing $n - 1$ inputs, the resultant $A \to A$ function is automatic.

Semiautomatic structures

Example

$(\mathbb{N}, +, <, =; \cdot)$ is not automatic.
For any constant n, implement multiplication by n as repeated addition.

Set theory

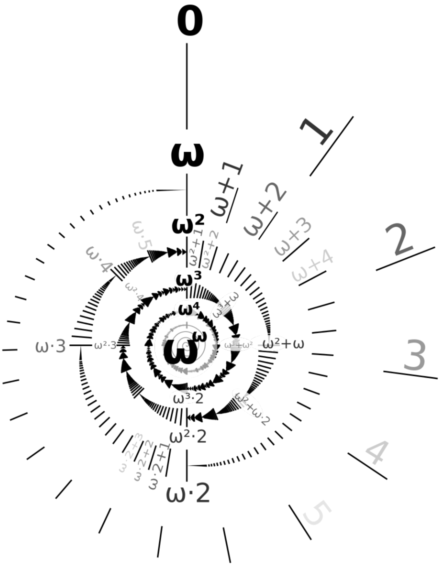
Ordinals

Equivalence classes of well-order, where a well-order $(S, <)$ obeys trichotomy, transitivity and well-foundedness (every nonempty subset has a minimum).

Intuitively, measures how many times a discrete process is repeated (possibly transfinitely many). For example,
$0, 1, 2, 3, \dots, \omega, \omega + 1, \dots, \omega + \omega = \omega \cdot 2,$
$\omega \cdot 2 + 1, \omega \cdot 2 + 2, \dots, \omega \cdot 2 + \omega = \omega \cdot 3, \dots, \omega \cdot 4, \dots,$
$\omega \cdot \omega = \omega^2, \dots, \omega^3, \dots, \omega^\omega, \dots.$

One way to generalise sum and products to infinite structures.

Ordinals



Semiautomatic Ordinal Structures

Semiautomatic ordinals with automatic addition

Existing characterisation

Theorem (Delhommé)

For any ordinal α the structure $(\alpha, +, <)$ is automatic iff $\alpha < \omega^\omega$.

Proof Sketch

Consider $\alpha = \omega^n$, any $\beta \in \alpha$ is of the form

$$\omega^{n-1} \cdot c_{n-1} + \cdots + \omega \cdot c_1 + c_0$$

for coefficients $c_0, \ldots, c_{n-1} \in \mathbb{N}$.
We can define $+$ on the n-ary convolution of an automatic copy of $(\mathbb{N}, +, <)$.

Incorporating semiautomaticity

Observation. Using the same representation, when $\beta \in \alpha$ is fixed, we can define left and right-multiplication by β in an automatic manner. (Addition and multiplication on ordinals are not commutative)

Theorem

For any ordinal $\alpha < \omega^\omega$ the structure $(\alpha, +, <, =; \cdot)$ is semiautomatic.

Left multiplication

Let

$$\beta = \omega^k \cdot b_k + \omega^{k-1} \cdot b_{k-1} + \cdots + \omega \cdot b_1 + b_0$$
$$\gamma = \omega^l \cdot c_l + \omega^{l-1} \cdot c_{l-1} + \cdots + \omega \cdot c_1 + c_0$$

expanding the giant expression, we get

$$\begin{aligned}\beta \cdot \gamma &= \beta \cdot \omega^l \cdot c_l + \beta \cdot \omega^{l-1} \cdot c_{l-1} + \cdots + \beta \cdot \omega \cdot c_1 + \beta \cdot c_0 \\ &\cdots \\ &= \omega^{k+l} \cdot c_l + \omega^{k+l-1} \cdot c_{l-1} + \cdots + \omega^{k+1} \cdot c_1 \\ &\quad + \left(\omega^k \cdot (b_k \cdot c_0) + \omega^{k-1} \cdot b_{k-1} + \cdots + \omega^{b_1} + b_0\right) \cdot 1_{c_0 \neq 0}\end{aligned}$$

where $1_{c_0 \neq 0}$ is 1 is $c_0 \neq 0$ and 0 otherwise.

Right multiplication

Ordinal multiplication distributes on the right, so we get a finite composition of

- right-multiplication by ω,
- right-multiplication by fixed constants,
- ordinal addition.

Right multiplication by ω

$$(\omega^3 \cdot b_3 + \omega^2 \cdot b_2 + \omega \cdot b_1 + b_0) \cdot \omega$$

$$= \begin{cases} \omega^4 & \text{if } b_3 > 0 \\ \omega^3 & \text{if } b_3 = 0, b_2 > 0 \\ \omega^2 & \text{if } b_3 = 0, b_2 = 0, b_1 > 0 \\ \omega & \text{if } b_3 = 0, b_2 = 0, b_1 = 0, b_0 > 0 \\ 0 & \text{otherwise} \end{cases}$$

Semiautomatic ordinals at ω^ω and beyond

Overview

Theorems (Jain, Khoussainov, Stephan, Teng and Zou)

- Let α be any countable ordinal, the structure $(\omega^\alpha; +, <, =)$ is semiautomatic.
- The semiring of polynomials over $\mathbb{N}$ $(\mathbb{N}[x]; +, \cdot, =)$ is semiautomatic.

Polynomials over $\mathbb{N}$

- Fix an semiautomatic copy A of $(\mathbb{N}, +, <; \cdot)$, add new "connectives" $\oplus, \otimes$
- Represent polynomials as lists of coefficients from A
- Represent elements as polynomials with connectives between them
  - $1, 10, 2 \oplus 0, 4$ represents the expression $(2x^2 + 10x + 1) \cdot (4x)$
- Define the quotient map val sending an expression to the canonical representation.
- val is not automatic, but for any polynomial $p \in \mathbb{N}[x]$, there is an automatic fragment of val that is "good enough".

Polynomials over ω

- We need even more connectives $\oplus_l, \oplus_r, \otimes_l, \otimes_r$
- For arbitrarily large $k \in \mathbb{N}$, $k + \omega = \omega$ and $\omega \cdot k \cdot \omega = \omega \cdot \omega$
- Refine error conditions to deal with this

Conclusion

The structure $(\omega^\omega; +, <, \cdot, =)$ is semiautomatic.

Semiautomatic Ring Structures

Overview

Rings are obtained by adding to an Abelian group a notion of multiplication.

Theorem (Jain, Khoussainov, Stephan, Teng and Zou)

For any $n \in \mathbb{N}$ the ring $(\mathbb{Z}(\sqrt{n}), \mathbb{Z}, +, <, =; \cdot)$ is semiautomatic.

Illustrating square roots

We illustrate with the simplest case, let $u = \dfrac{1+\sqrt{5}}{2}$ denote the golden ratio.

Theorem (Jain, Khoussainov, Stephan, Teng and Zou)

$(\mathbb{Z}[u], +, <, =; \cdot)$ is semiautomatic.

The ingredients

- $3 = u^{-2} + u^2$ for any $x + yu \in \mathbb{Z}[u]$, so update coefficients until

$$x + yu = \sum_i a_i u^i$$

  but now each $|a_i| \leq 2$.
- Tail bound – $\sum_{i \leq 2} u^i$ is a geometric series

Sign test

- To check if $a + b = c$ we perform sign test on $a + b - c$.
- Given a list of coefficients, we slide over them and use the equation $u^2 = u + 1$ to update our memory
- The first time any coefficient overshoot $4 \cdot \sum_{i \leq 2} u^i$, by our earlier tail bound we will know the sign
- If all input is processed, do a finite case distinction
- Use the characterisation that automatic functions are also computable by linear-time one-tape Turing machines where input and output start at the same position (Case, Jain, Seah and Stephan) to look through coefficients in order.

Generalising to cube roots

The case of $\sqrt[3]{7}$

Let $u^{-1} = 2 - \sqrt[3]{7}$ and we use a similar strategy to show $(\mathbb{Z}[u], +, <, =; \cdot)$ is semiautomatic.

- To achieve our coefficient bound we have

$$u^3 - 12u^2 + 6u - 1 = 0$$

  where 12 dominates.

- The tail bound is also geometric, but since we update two coefficients at once in every step of the sign test, we need more refined bounds.

The case of $\sqrt[3]{7}$

- Denote our current coefficients a_i, a_{i+1}, a_{i+2} and we terminate if any condition here is broken
  - $|a_{i+2}| \leq 16k'$,
  - $|a_{i+1}| \leq 4k'$, or
  - $|a_i| \leq k'$, or

  where $k' = 360$.

This gives us a semiautomatic ring containing $\sqrt[3]{7}$.

Conclusion

Contributions

- Incorporate notion of semiautomaticity into small ordinals (below ω^ω).
- Found semiautomatic rings containing cube roots, generalising existing result.

Future directions

On ordinals

Question

For how large α can we find semiautomatic representations of $(\alpha; +, <, \cdot, =)$?

Future directions

On semiautomatic rings

Question

For every algebraic number c is there a semiautomatic ring with order which contains c?

Question

Is there a semiautomatic ring with order that simultaneously contains $\sqrt{2}$, $\sqrt{3}$ and $\frac{1}{2}$?
When these 3 numbers are simultaneously present, the ring has enough information to encode the angle 15 degrees.