MA1100 Fundamental Concepts of Mathematics AY2017/18 Sem 1

Homework 5

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Q 1. Let A be a finite set of size m where $m \geqslant 1$, and let a be an element of A. Prove that one has $|A \setminus \{a\}| + 1 = m$.

Proof. A is finite, so $\{a\} \subseteq A$ is also finite, by complement principle, $|A \setminus \{a\}| + |\{a\}| = |A|$, so $|A \setminus \{a\}| + 1 = m$. □

Q 2. Let S be a finite set and let $f : S \to S$ be a function. Prove that f is injective iff f is surjective.

Proof. Suppose $f : S \to S$ is injective. For any subset $X \subseteq S$, let $f(X) \subseteq S$ be the X-image of f,

$$f(X) := \{ y \in S : \exists x \in X.\ f(x) = y \}.$$

Clearly $|f(S)| \leqslant |S|$ and $|f(S)|$ is finite. Since f is injective, by injection principle, $|S| \leqslant |f(S)|$, so $|f(S)| = |S|$. By corollary of complement principle, $f(S) = S$ and f is surjective. Conversely suppose f is surjective. For any subset $Y \subseteq S$, let $f^*(Y) \subseteq S$ denote the Y-pre-image of f

$$f^*(Y) := \{ x \in S : f(x) \in Y \}.$$

Clearly $|f^*(S)| \leqslant |S|$, and since f is surjective, for every $y \in S$, $f^*(\{y\})$ is non-empty.

$$\forall y \in S.\ |f^*(\{y\})| \geqslant 1$$

Because f is well-defined, for any distinct pair of elements in S, the f-preimage of their singletons are disjoint.

$$\forall y_1, y_2 \in S.\ y_1 \neq y_2 \implies f^*(\{y_1\}) \cap f^*(\{y_2\}) = \emptyset$$

Because f is totally-defined, the union of f-preimages of every element in its range will cover the domain S, so let $|S|$ be n, and for $i \in \{1, 2, \ldots, n\}$, let y_i denote each element in S,

$$\bigcup_{i=1}^{n} f^*(\{y_i\}) = S$$

$$\left| \bigcup_{i=1}^{n} f^*(\{y\}) \right| = |S|$$

$$\sum_{i=1}^{n} |f^*(\{y_i\})| = n$$

for each y_i, $f^*(\{y_i\})$ is non-empty

$$1 \cdot n \leqslant \sum_{i=1}^{n} |f^*(\{y_i\})| = n$$

this means that for each $y_i \in S$, $|f^*(\{y_i\})| = 1$, therefore f is injective. □

Q 3. Let $m, n \in \mathbb{N}$ be so that $n > m$. Prove that there is no injective function f from $\{1, \ldots, n\}$ to $\{1, \ldots, m\}$. (*Pigeonhole Principle*)

Proof. First note that $\{1, \ldots, n\} \cong \mathbb{N}_{<n}$ and $\{1, \ldots, m\} \cong \mathbb{N}_{<m}$ are finite,

$$n > m$$
$$|\{1, \ldots, n\}| > |\{1, \ldots, m\}|$$

Then by (contrapositive of) injection principle, there does not exist injective map f from $\{1, \ldots, n\}$ to $\{1, \ldots, m\}$. $\square$

Q 4. Prove that the function $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(n) := \begin{cases} \frac{n-1}{2}; & \text{if } n \text{ is odd,} \\ \frac{-n}{2}; & \text{if } n \text{ is even,} \end{cases}$ is bijective.
($\mathbb{N}$ *starts from 1 in this question.*)

Proof. Define $g : \mathbb{Z} \to \mathbb{N}, z \mapsto \begin{cases} -2z; & \text{if } z < 0, \\ 2z+1; & \text{if } z \geqslant 0. \end{cases}$
For any odd $n \in \mathbb{N}$,

$$(g \circ f)(n) = g\left(\frac{n-1}{2}\right) = 2\left(\frac{n-1}{2}\right) + 1 = n,$$

and for any even $n \in \mathbb{N}$,

$$(g \circ f)(n) = g\left(\frac{-n}{2}\right) = -2\left(\frac{-n}{2}\right) = n.$$

So $g \circ f = \mathrm{id}_{\mathbb{N}}$.
For any $z \in \mathbb{Z}, z < 0$, $-2z > 0$ and is even,

$$(f \circ g)(z) = f(-2z) = \frac{-(-2z)}{2} = z,$$

and when $z \geqslant 0$, $2z + 1 > 0$ and is odd,

$$(f \circ g)(z) = f(2z+1) = \frac{(2z+1)-1}{2} = z.$$

So $f \circ g = \mathrm{id}_{\mathbb{Z}}$. Since f is invertible, f is bijective. $\square$

Q 5. Let F be a finite set and let I be an infinite set. Prove that $I \setminus F$ is infinite.

Proof. Without loss of generality, suppose $F \subseteq I$, then $I = F \cup (I \setminus F)$. (Otherwise consider the intersection of F and I.) Suppose for a contradiction $I \setminus F$ is finite, since $I \setminus F$ and F are finite and disjoint, by addition principle,

$$|F| + |I \setminus F| = |F \cup (I \setminus F)|$$

and $F \cup (I \setminus F)$ is also finite. But $F \cup (I \setminus F) = I$, so id_I is a bijective map from an infinite set to a finite set, a contradiction. $\square$

Q 6. Let S be a set. Prove that S is countable iff there is an injective function $f : S \to \mathbb{N}$.

Proof. If S is countable, $S \preccurlyeq \mathbb{N} \iff$ exists injective function $f : S \to \mathbb{N}$. $\square$

Q 7. Let S be a non-empty set. Prove that S is countable iff there is a surjective map $f : \mathbb{N} \to S$.

Proof. If S is countable, $S \preccurlyeq \mathbb{N} \iff$ exists injective map $g : S \to \mathbb{N} \iff$ exists surjective map $f : \mathbb{N} \to S$ (consequence of Axiom of Choice, because $S \neq \emptyset$). $\square$

Q 8. Prove that if $C_1, \dots, C_n$ is countable, then $C_1 \times C_2 \times \cdots \times C_n$ is countable.

Lemma 8.1. $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$. *(Another proof in Q11)*

Proof. Consider this visual representation of $\mathbb{N} \times \mathbb{N}$

$$\begin{array}{cccccc}
\mathbb{N} & 0 & 1 & 2 & 3 & \dots \\
0 & (0,0) & (0,1) & (0,2) & (0,3) & \dots \\
1 & (1,0) & (1,1) & (1,2) & (1,3) & \dots \\
2 & (2,0) & (2,1) & (2,2) & (2,3) & \dots \\
3 & (3,0) & (3,1) & (3,2) & (3,3) & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Define a bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by diagonally tracing the diagram above, ie $f(0) := (0,0), f(1) := (0,1), f(2) := (1,0), f(3) := (0,2), f(4) := (1,1), f(5) := (2,0), \dots$. $\square$

Lemma 8.2. *Product of two countable sets is countable.*

Proof. Suppose C_1, C_2 are countable sets, $C_1 \preccurlyeq \mathbb{N}$ and $C_2 \preccurlyeq \mathbb{N}$, so there exists injective maps $f : C_1 \to \mathbb{N}$ and $g : C_2 \to \mathbb{N}$, then define h as

$$\begin{aligned} h : C_1 \times C_2 &\to \mathbb{N} \times \mathbb{N}, \\ (c_1, c_2) &\mapsto (f(c_1), g(c_2)). \end{aligned}$$

Suppose $c_1, c_1' \in C_1$ and $c_2, c_2' \in C_2$ such that $h(c_1, c_2) = h(c_1', c_2')$, then $(f(c_1), g(c_2)) = (f(c_1'), g(c_2'))$ which means $f(c_1) = f(c_1')$ and $g(c_2) = g(c_2')$, and because f and g are injective, $c_1 = c_1'$ and $c_2 = c_2'$, so $(c_1, c_2) = (c_1', c_2')$ and h is injective. This means $C_1 \times C_2 \preccurlyeq \mathbb{N} \times \mathbb{N}$, and because $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$ (from Lemma 8.1), $C_1 \times C_2 \preccurlyeq \mathbb{N}$. $\square$

Proposition. *Product of finitely many countable sets is countable.*

Proof. Product of 2 countable sets is countable. Now suppose the product of n countable sets, $C_1 \times C_2 \times \cdots \times C_n$, is countable, $C_1 \times C_2 \times \cdots \times C_n \preccurlyeq \mathbb{N}$, and C_{n+1} is also countable. Then by Lemma 8.2, $(C_1 \times C_2 \times \cdots \times C_n) \times C_{n+1} \preccurlyeq \mathbb{N}$. Therefore by induction, for any $n \in \mathbb{N}$, $n \geqslant 2$, $C_1 \times C_2 \times \cdots \times C_n$ is countable. $\square$

Q 9. Let X and Y be any two sets. Suppose $|X| = |Y|$. Show that $|\mathcal{P}(X)| = |\mathcal{P}(Y)|$.

Proof. Suppose X and Y are any two sets where $|X| = |Y|$, then there exists a bijective map $f : X \to Y$. For any $C \subseteq X$, the f-image of C is a subset of Y where

$$f(C) := \{\, y \in Y : \exists c \in C.\ f(c) = y \,\}$$

and because f is bijective, $f(C)$ is uniquely determined by C.
Similarly, for any $D \subseteq Y$, the f-preimage of D is a subset of X where

$$f^*(D) := \{\, x \in X : f(x) \in D \,\}$$

which is also uniquely determined by D because f is bijective.
We can define the bijective map ψ

$$\psi : \mathcal{P}(X) \to \mathcal{P}(Y), \quad C \mapsto f(C).$$

For any $C, C' \in \mathcal{P}(X)$, if $C \neq C'$, then because f is bijective, $f(C) \neq f(C')$, so ψ is injective. For any $D \in \mathcal{P}(Y)$, because f is surjective, $f^*(D) \subseteq X$, so in particular, there exists $C \in \mathcal{P}(X)$ where $f(C) = D$, so ψ is surjective.
Hence ψ is bijective, and as a result $|\mathcal{P}(X)| = |\mathcal{P}(Y)|$ $\square$

Definition. For any sets A and B, let $\mathrm{Maps}(A, B)$ denote the subset of $A \times B$ defined by

$$\mathrm{Maps}(A, B) := \left\{ \begin{array}{ll} \varphi \in \mathcal{P}(A \times B) : & \varphi \text{ as a relation from } A \text{ to } B \\ & \text{is totally defined and well-defined} \end{array} \right\}$$

Q 10. Let X and Y be any two sets, and consider the set $\mathrm{Maps}(X, Y)$ of all maps from X to Y. Show that $|\,\mathrm{Maps}(X, Y)| \leqslant |\mathcal{P}(X \times Y)|$.

Proof. Since by definition, $\mathrm{Maps}(X, Y) \subseteq \mathcal{P}(X \times Y)$, define the map

$$\begin{aligned} \Phi : \mathrm{Maps}(X, Y) &\to \mathcal{P}(X \times Y) \\ \varphi &\mapsto \varphi \end{aligned}$$

which is almost the identity map, and is clearly injective. So $|\,\mathrm{Maps}(X, Y)| \leqslant |\mathcal{P}(X \times Y)|$. $\square$

Q 11. Use the unique prime factorisation property of $\mathbb{Z}$ (fundamental theorem of arithmetic) and the Schröder-Bernstein theorem to show that $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.

Proof. The map $\varphi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}, n \mapsto (12, n)$ is clearly an injective map from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$, so $\mathbb{N} \preccurlyeq \mathbb{N} \times \mathbb{N}$. Now consider the map ψ,

$$\begin{aligned} \psi : \mathbb{N} \times \mathbb{N} &\to \mathbb{N}, \\ (a, b) &\mapsto 2^a \cdot 3^b \end{aligned}$$

For any $a, b, c, d \in \mathbb{N}$ where $\psi(a, b) = \psi(c, d)$, $2^a 3^b = 2^c 3^d$. Then by uniqueness of prime factors, $a = c$ and $b = d$, so $(a, b) = (c, d)$, and ψ is injective. Therefore $\mathbb{N} \times \mathbb{N} \preccurlyeq \mathbb{N}$.
By Schröder-Bernstein theorem, $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$. $\square$

Q 12. Show that

$$|\mathcal{P}(\mathbb{N})| \leqslant |\operatorname{Maps}(\mathbb{N}, \mathbb{N})|.$$

Use this and the above results to deduce that

$$|\mathcal{P}(\mathbb{N})| = |\operatorname{Maps}(\mathbb{N}, \mathbb{N})|.$$

Proof. For any $S \subseteq \mathbb{N}$, define Ψ,

$$\begin{aligned} \Psi : \mathcal{P}(\mathbb{N}) &\to \operatorname{Maps}(\mathbb{N}, \mathbb{N}), \\ S &\mapsto \lambda_S, \\ \text{where } \lambda_S : \mathbb{N} &\to \mathbb{N}, \\ n &\mapsto \begin{cases} 1 & \text{if } n \in S, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

For any two subsets of $\mathbb{N}$, $S_1, S_2 \in \mathcal{P}(\mathbb{N})$, if $S_1 \neq S_2$, without loss of generality, $\exists u \in S_1.\ u \notin S_2$. Then $\Psi(S_1)(u) = 1 \neq 0 = \Psi(S_2)(u)$. So in particular, $\Psi(S_1) \neq \Psi(S_2)$. Hence Ψ is injective and $|\mathcal{P}(\mathbb{N})| \leqslant |\operatorname{Maps}(\mathbb{N}, \mathbb{N})|$.

$$\begin{aligned} \text{From Q11,} &\quad & \mathbb{N} &\cong \mathbb{N} \times \mathbb{N} \\ \text{from Q9,} &\quad & \mathcal{P}(\mathbb{N}) &\cong \mathcal{P}(\mathbb{N} \times \mathbb{N}) \\ \text{from Q10,} &\quad & \operatorname{Maps}(\mathbb{N}, \mathbb{N}) &\preccurlyeq \mathcal{P}(\mathbb{N} \times \mathbb{N}) \\ \text{therefore} &\quad & \operatorname{Maps}(\mathbb{N}, \mathbb{N}) &\preccurlyeq \mathcal{P}(\mathbb{N}) \end{aligned}$$

Then because $\mathcal{P}(\mathbb{N}) \preccurlyeq \operatorname{Maps}(\mathbb{N}, \mathbb{N})$ and $\operatorname{Maps}(\mathbb{N}, \mathbb{N}) \preccurlyeq \mathcal{P}(\mathbb{N})$, by Schröder-Bernstein theorem, $\mathcal{P}(\mathbb{N}) \cong \operatorname{Maps}(\mathbb{N}, \mathbb{N})$. $\square$