MA2101S Homework 8

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1 Question 1

(a) Characteristic polynomial of T is
$$-(t-2)^5(t-3)^2$$

For eigenvalue $\lambda = 2$,

(b) $\dim(E_2) = 2$ and $\dim(K_2) = 5$
(c) smallest $p = 3$
(d) $\dim(\operatorname{Ker}(T|_{K_2} - 2)) = 2$, $\dim(\operatorname{Ker}(T|_{K_2} - 2)^2) = 4$ and $\dim(\operatorname{Ker}(T|_{K_2} - 2)^3) = 5$.

For eigenvalue $\lambda = 3$,

(b) $\dim(E_3) = 2$ and $\dim(K_3) = 2$
(c) smallest $p = 1$
(d) $\dim(\operatorname{Ker}(T|_{K_3} - 2)) = 2$, $\dim(\operatorname{Ker}(T|_{K_3} - 2)^2) = 2$ and $\dim(\operatorname{Ker}(T|_{K_3} - 2)^3) = 2$. $\square$

2 Question 2

(a) First, find characteristic polynomial of A,

$$\begin{aligned}\det(A - tI) &= \begin{vmatrix} 11-t & -4 & -5 \\ 21 & -8-t & -11 \\ 3 & -1 & -t \end{vmatrix} \\ &= -t^3 + 3t^2 - 4 \\ &= -(t-2)^2(t+1)\end{aligned}$$

A has eigenvalues 2 and -1.

- For eigenvalue 2,

$$A - 2I = \begin{pmatrix} 9 & -4 & -5 \\ 21 & -10 & -11 \\ 3 & -1 & -2 \end{pmatrix}$$

has kernel $\operatorname{span}\{\, (1,1,1)^t \,\}$. The other basis vector for K_2 can be found by solving for x such that $(A-2I)x = (1,1,1)^t$. By Gauss-Jordan elimination

$$\left(\begin{array}{ccc|c} 9 & -4 & -5 & 1 \\ 21 & -10 & -11 & 1 \\ 3 & -1 & -2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

It can be verified that $(A-2I)(1,2,0)^t$ is indeed $(1,1,1)$, so $(A-2I)^2(1,2,0)^t = 0$. A basis for K_2 is $\{\, (1,1,1)^t, (1,2,0)^t \,\}$.

- For eigenvalue -1,

$$A + I = \begin{pmatrix} 12 & -4 & -5 \\ 21 & -7 & -11 \\ 3 & -1 & 1 \end{pmatrix}$$

which has kernel $\operatorname{span}\{\, (1,3,0)^t \,\}$.

Let Q be given by

$$Q := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 0 & 0 \end{pmatrix}$$

which clearly has linearly independent columns, then from computations above,

$$\begin{aligned} AQ &= \left(\begin{array}{c|c|c} 2\begin{pmatrix}1\\1\\1\end{pmatrix} & \begin{pmatrix}1\\1\\1\end{pmatrix} + 2\begin{pmatrix}1\\2\\0\end{pmatrix} & -1\begin{pmatrix}1\\3\\0\end{pmatrix} \end{array}\right) \\ &= Q\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

so by setting J as the matrix in Jordan canonical form shown above, we have $Q^{-1}AQ = J$. $\square$

(b) Find the characteristic polynomial of A,

$$\begin{aligned}\det(A-tI) &= \begin{vmatrix} 2-t & 1 & 0 & 0 \\ 0 & 2-t & 1 & 0 \\ 0 & 0 & 3-t & 0 \\ 0 & 1 & -1 & 3-t \end{vmatrix} \\ &= (t-2)^2(t-3)^2\end{aligned}$$

A has eigenvalues 2 and 3.

- For eigenvalue 2,

$$A-2I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

which has kernel given by $\operatorname{span}\{\,(1,0,0,0)^t\,\}$. Using the same shortcut, solve for x in $(A-2I)x = (1,0,0,0)^t$, by Gauss-Jordan elimination,

$$\left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

A trivial verification shows that $(A-2I)(0,1,0,-1)^t = (1,0,0,0)^t$ indeed, so $\{\,(1,0,0,0)^t, (0,1,0,-1)^t\,\}$ forms a basis for K_2.

- For eigenvalue 3,

$$A-3I = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

which has a kernel $\operatorname{span}\{\,(0,0,0,1)^t, (1,1,1,0)^t\,\}$.

Let Q be given by

$$Q := \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix}$$

a trivial computation shows that it has linearly independent columns, then from the computations above

$$AQ = \left(\begin{array}{c|c|c|c} 2\begin{pmatrix}1\\0\\0\\0\end{pmatrix} & \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} & 3\begin{pmatrix}0\\0\\0\\1\end{pmatrix} & 3\begin{pmatrix}1\\1\\1\\0\end{pmatrix} \end{array} \right)$$
$$= Q\begin{pmatrix}2&1&0&0\\0&2&0&0\\0&0&3&0\\0&0&0&3\end{pmatrix}$$

so by setting J as the matrix in Jordan canonical form shown above, we have $Q^{-1}AQ = J$. □

3 Question 3

(a) Let $\mathcal{B} := (e^t, te^t, t^2e^t, t^3e^t, e^{3t}, te^{3t})$, to compute $[T]_{\mathcal{B}}$, first find out where T sends the basis to,

$$\begin{aligned} T(e^t) &= e^t \\ T(te^t) &= e^t + te^t \\ T(t^2e^t) &= 2te^t + t^2e^t \\ T(t^3e^t) &= 3t^2e^t + t^3e^t \\ T(e^{3t}) &= 3e^{3t} \\ T(te^{3t}) &= e^{3t} + 3te^{3t} \end{aligned}$$

which is enough information to consolidate the matrix representation of T with respect to $\mathcal{B}$,

$$[T]_{\mathcal{B}} = \begin{pmatrix}1&1&0&0&0&0\\0&1&2&0&0&0\\0&0&1&3&0&0\\0&0&0&1&0&0\\0&0&0&0&3&1\\0&0&0&0&0&3\end{pmatrix}.$$

$[T]_{\mathcal{B}}$ is *almost* in Jordan canonical form, we just need to find vectors $c, d \in V$ such that $(T-I)c = te^t$ and $(T-I)d = c$, which are as given

$$T(\frac{1}{2}t^2e^t) = \frac{1}{2}t^2e^t + te^t$$
$$T(\frac{1}{6}t^3e^t) = \frac{1}{6}t^3e^t + \frac{1}{2}t^2e^t$$

then it becomes clear that with respect to a new ordered basis $\mathcal{B}' := (e^t, te^t, \frac{1}{2}t^2e^t, \frac{1}{6}t^3e^t, e^{3t}, te^{3t})$,

$$[T]_{\mathcal{B}'} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}. \qquad \square$$

(b) Let $\mathcal{B} := \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$ be an ordered basis for V. First find out where T sends this basis to

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$
$$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ -1 & 0 \end{pmatrix}$$
$$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 3 & 0 \end{pmatrix}$$
$$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$$

this is enough information to find $[T]_{\mathcal{B}}$

$$[T]_{\mathcal{B}} = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Proceed to decompose T into Jordan form (if possible), the characteristic polynomial is

$$\begin{aligned}\det([T]_{\mathcal{B}} - tI) &= \begin{vmatrix} 2-t & 0 & 1 & 0 \\ 0 & 3-t & -1 & 1 \\ 0 & -1 & 3-t & 0 \\ 0 & 0 & 0 & 2-t \end{vmatrix} \\ &= (2-t)^2 \begin{vmatrix} 3-t & -1 \\ -1 & 3-t \end{vmatrix} \\ &= (2-t)^2 \left[(3-t)^2 - 1\right] \\ &= (t-2)^3(t-4)\end{aligned}$$

For now, take all column vectors with respect to ordered basis $\mathcal{B}$.

- For eigenvalue 4,

  $$[T]_{\mathcal{B}} - 4I = \begin{pmatrix} -2 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

  which has kernel given by $\operatorname{span}\left\{ (1, -2, 2, 0)^t \right\}$.

- For eigenvalue 2,

  $$[T]_{\mathcal{B}} - 2I = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

  observe that $\ker([T]_{\mathcal{B}} - 2I) = \operatorname{span}\left\{ (1, 0, 0, 0)^t \right\}$.

  Solve for $x \in V$ such that $([T]_{\mathcal{B}} - 2I)[x]_{\mathcal{B}} = (1, 0, 0, 0)^t$, by Gauss-Jordan elimination

  $$\left(\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

  we have $([T]_{\mathcal{B}} - 2I)(0, 1, 1, 0)^t = (1, 0, 0, 0)^t$.

Next, solve for $y \in V$ such that $([T]_{\mathcal{B}} - 2I)[y]_{\mathcal{B}} = (0,1,1,0)^t$, by Gauss-Jordan elimination

$$\left(\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

so we have $([T]_{\mathcal{B}} - 2I)(0,-1,0,2)^t = (0,1,1,0)^t$.

Then from the computations above, we see that with respect to the new ordered basis $\mathcal{B}' := \left(\begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix}\right)$,

$$[T]_{\mathcal{B}'} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}. \qquad \square$$

4 Question 4

For any $r \in \mathbb{N}$, let $P_r \in \mathbb{M}_r(K)$ denote the $r \times r$ matrix

$$P_r := \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & ⋰ & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}.$$

It can easily be verified that $P_r P_r = I_r$. Observe that post-multiplication by P_r reverses the columns, while pre-multiplication by P_r will reverse the rows, so $A^t = P_n A P_n$. A and A^t having the same Jordan form would be a simple corollary. $\square$

(Did I just defeat the point of this question?)

5 Question 4 (again)

It is obvious that A and A^t has the same characteristic polynomial, and hence the same eigenvalues. For any eigenvalue λ of A and A^t, observe that $(A - \lambda I)^t = A^t - \lambda I$. For any $r \in \mathbb{Z}_{>0}$, $\left((A - \lambda I)^r\right)^t = \left((A - \lambda I)^t\right)^r = (A^t - \lambda I)^r$. Then because row rank is the same as column rank, $(A - \lambda I)^r$ and $(A^t - \lambda I)^r$ have the same rank. From this we can conclude that for each eigenvalue λ, A and A^t have the same associated dot diagrams, hence the same Jordan blocks. Therefore A and A^t have the same Jordan form.

So we have $\exists Q, R \in \mathrm{GL}_n(K), J \in \mathbb{M}_n(K)$ with J in Jordan form such that $J = QAQ^{-1} = RA^tR^{-1}$. Then $A = Q^{-1}RA^tR^{-1}Q$, which shows that $A \sim A^t$. □

6 Question 5

(a) Let $N := A - \lambda I$ in $\mathbb{M}_n(K)$. For any $r \in \mathbb{N}$, computation shows that

$$N^r(i,j) = \delta(i+r,j) = \begin{cases} 1 & \text{if } i+r=j, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Let $D := \lambda I$, then $A = D+N$, note that $DN = ND$. Since they commute, the binomial theorem applies, then for any $r \in \mathbb{N}$ with $r \geqslant n$,

$$\begin{aligned} A^r &= (D+N)^r \\ &= \sum_{k=0}^{r} \binom{r}{k} D^{r-k}N^k \\ &= \sum_{k=0}^{n-1} \binom{r}{k} \lambda^{r-k}N^k \\ A^r(i,j) &= \sum_{k=0}^{n-1} \binom{r}{k} \lambda^{r-k}N^k(i,j) \\ &= \sum_{k=0}^{n-1} \binom{r}{k} \lambda^{r-k}\delta(i+k,j) \end{aligned}$$

the Kronecker delta reduces the sum to a single term if $i \leqslant j$,

$$A^r(i,j) = \begin{cases} \binom{r}{k} \lambda^{r-k} & \text{if } \exists k \in \mathbb{N}.\, i+k=j, \\ 0 & \text{otherwise.} \end{cases}$$ □

7 Question 6

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \in \mathbb{M}_n(\mathbb{F}_p)$$

(a) The characteristic polynomial of A is

$$\begin{aligned}
\det(A - tI) &= \begin{vmatrix} 1-t & 1 & \cdots & 1 \\ 1 & 1-t & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-t \end{vmatrix} \\
&\Big\downarrow \text{add row } 2, \dots, n \text{ to row } 1 \\
&= \begin{vmatrix} n-t & n-t & \cdots & n-t \\ 1 & 1-t & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-t \end{vmatrix} \\
&= \begin{vmatrix} n-t & 1 & \cdots & 1 \\ n-t & 1-t & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ n-t & 1 & \cdots & 1-t \end{vmatrix} \\
&\Big\downarrow \text{subtract row } 1 \text{ from row } 2, \dots, n \\
&= \begin{vmatrix} n-t & 1 & 1 & \cdots & 1 \\ 0 & -t & 0 & \cdots & 0 \\ 0 & 0 & -t & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -t \end{vmatrix} \\
&= (n-t)(-t)^{n-1} \\
&= (-1)^n (t^n - nt^{n-1})
\end{aligned}$$

□

(b) From (a), A has characteristic polynomial $(-1)^n t^{n-1}(t-n)$. So A has eigenvalues 0 and n ($n \neq 0$ as $p \nmid n$).

- For eigenvalue n, by inspection,

$$A \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} n \\ n \\ \vdots \\ n \end{pmatrix}$$
$$= n \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

  so $K_n = E_n$ has a basis $\operatorname{span}\{ (1, 1, \dots, 1)^t \}$.
- For eigenvalue 0, observe that $\operatorname{rank}(A) = 1$, so $\operatorname{nullity}(A) = n - 1$. So A is in fact diagonalisable. By further observation, these $n-1$ linearly independent vectors form the basis for $\ker(A)$,

$$\{ -e_1 + e_j : j \in \{ 2, \dots, n \} \} .$$

Define $Q \in \mathbb{M}_n(\mathbb{F}_p)$ as

$$Q := \begin{pmatrix} 1 & -1 & -1 & \cdots & -1 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

As vectors from different eigen-bases are linearly independent, Q is invertible, then from computations above,

$$AQ = \begin{pmatrix} n & 0 & \cdots & 0 \\ n & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ n & 0 & \cdots & 0 \end{pmatrix}$$
$$= Q \begin{pmatrix} n & \\ & \bigcirc \end{pmatrix}$$

so by setting J as the diagonal matrix obtained above, we have $Q^{-1}AQ = J$. $\square$