MA2104 Assignment 3

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Question 1

> Find the point on the paraboloid $z = \dfrac{x^2}{4} + \dfrac{y^2}{25}$ that is closest to the point $(3, 0, 0)$.

Solution. For any point (x, y, z), the distance between (x, y, z) and $(3, 0, 0)$ is given by

$$D(x, y, z) := \sqrt{(x-3)^2 + y^2 + z^2}.$$

By hint, D is minimum if and only if D^2 is minimum, so it suffice to find minimum of D^2 given the constraint that

$$g(x, y, z) := \frac{x^2}{4} + \frac{y^2}{25} - z = 0.$$

Proceed to use Lagrange Multipliers to find maximum of D^2 given $g(x, y, z) = 0$, since $\nabla g \neq \mathbf{0}$,

$$\begin{aligned} \nabla(D^2)(x, y, z) &= \langle 2(x-3), 2y, 2z \rangle \\ \nabla g(x, y, z) &= \left\langle \frac{x}{2}, \frac{2y}{25}, -1 \right\rangle \end{aligned}$$

Now proceed to solve $\nabla(D^2)(x, y, z) = \lambda \nabla g(x, y, z)$, which gives the following system of equations

$$\left.\begin{aligned} 2(x-3) &= \lambda\frac{x}{2} \\ 2y &= \lambda\frac{2y}{25} \\ 2z &= -\lambda \\ \frac{x^2}{4} + \frac{y^2}{25} &= z \end{aligned}\right\} \implies \left.\begin{aligned} 4(x-3) &= \lambda x \\ 25y &= \lambda y \\ z &= -\frac{\lambda}{2} \\ \frac{x^2}{4} + \frac{y^2}{25} &= z \end{aligned}\right\}$$

From second equation, $y = 0$ or $\lambda = 25$.

- Case $\lambda = 25$, then $z = -\frac{25}{2}$, and

$$\begin{aligned} 4x - 12 &= 25x \\ x &= -\frac{4}{7} \end{aligned}$$

This has no solution in $\mathbb{R}$ as $\frac{y^2}{25} \geq 0 > z - \frac{x^2}{4} = -\frac{25}{2} - \frac{4}{49}$.

- Case $y = 0$, then fourth equation reduces to $x^2 = 4z$,

$$\begin{aligned} x^2 &= 4z \\ x^2 &= -2\lambda \\ \lambda &= -\frac{x^2}{2} \end{aligned}$$

substituting that into our first equation we get

$$\begin{aligned} 4x - 12 &= -\frac{x^3}{2} \\ x^3 + 8x - 24 &= 0 \\ x &= 2 \end{aligned}$$

Then we get $z = 1$. The only critical point is $(2, 0, 1)$. ■

Question 2

> Suppose that the temperature of a metal plate is given by $T(x, y) = x^2 + 2x + y^2$ for points (x, y) on the elliptical plate defined by $x^2 + 4y^2 \leq 24$.
>
> Find the maximum and minimum temperatures on the plate.

Solution. The gradient vector for T is given by

$$\nabla T(x, y) = \langle 2x + 2, 2y \rangle .$$

The critical points are when $\nabla T = \mathbf{0}$, so

$$\left.\begin{aligned} 2x + 2 &= 0 \\ 2y &= 0 \end{aligned}\right\}$$

The only critical point obtained is $(-1,0)$, which is eyeballed to be inside the elliptical plate.

Next, proceed to use Lagrange multipliers to find critical points on the boundary, let $g(x,y) := x^2+4y^2 = 24$ be our constraint. Then

$$\nabla g(x,y) = \langle 2x, 8y \rangle .$$

Solving $\nabla T(x,y) = \lambda \nabla g(x,y)$, we obtain the system of equations

$$\left.\begin{aligned} x+1 &= \lambda x \\ y &= 4\lambda y \\ x^2+4y^2 &= 24 \end{aligned}\right\}$$

From second equation, $\lambda = \frac{1}{4}$ or $y=0$,

- Case $\lambda = \frac{1}{4}$, then

$$\begin{aligned} x+1 &= \frac{x}{4} \\ x &= -\frac{4}{3} \end{aligned}$$

  Substituting that into our constraint,

$$\begin{aligned} \frac{16}{9} + 4y^2 &= 24 \\ y^2 &= \frac{50}{9} \\ y &= \pm\frac{5\sqrt{2}}{3} \end{aligned}$$

- Case $y=0$, then $x^2 = 24$, so $x = \pm 2\sqrt{6}$.

Tabulating the critical points,

(x,y)	$(-1,0)$	$(-\frac{4}{3}, \pm\frac{5\sqrt{2}}{3})$	$(2\sqrt{6},0)$	$(-2\sqrt{6},0)$
$T(x,y)$	-1	$\frac{14}{3}$	$24+4\sqrt{6}$	$24-4\sqrt{6}$
	min		max	

■

Question 3

> Evaluate the following integral
>
> $$\int_0^2 \int_{\sqrt{y}}^2 \sqrt{x^2 + y}\, dx\, dy.$$

Solution. Let the region of integration be called D, so

$$D = \{\, (x, y) : 0 \le y \le 4, \sqrt{y} \le x \le 2 \,\}.$$

But D can also be expressed as

$$D = \left\{\, (x, y) : 0 \le x \le 2, 0 \le y \le x^2 \,\right\}.$$

This allows us to rewrite the integral as

$$\begin{aligned}
\iint_D \sqrt{x^2 + y}\, dA &= \int_0^2 \int_0^{x^2} \sqrt{x^2 + y}\, dy\, dx \\
&= \int_0^2 \left[\frac{2}{3} \left(x^2 + y\right)^{3/2} \right]_0^{x^2} dx \\
&= \frac{2}{3} \int_0^2 \left((2x^2)^{3/2} - (x^2)^{3/2} \right) dx \\
&= \frac{2}{3} \int_0^2 \left(2^{3/2} - 1 \right) x^3\, dx \\
&= \frac{2}{3} \left(2^{3/2} - 1 \right) \left[\frac{x^4}{4} \right]_0^2 dx \\
&= \frac{8}{3} \left(2^{3/2} - 1 \right)
\end{aligned}$$

■

Question 4

> Rewrite the following iterated integral in the order $dy\, dx\, dz$:
>
> $$\int_{-1}^1 \int_0^{\sqrt{1-x^2}} \int_0^{y/2} f(x, y, z)\, dz\, dy\, dx.$$

Solution. Let D denote the region of integration, then it can be given by

$$D = \left\{ (x, y, z) : -1 \leq x \leq 1, 0 \leq y \leq \sqrt{1-x^2}, 0 \leq z \leq \frac{y}{2} \right\}.$$

To integrate in the order $dy\ dx\ dz$, first find absolute bounds for z.

$$z \leq \frac{y}{2} \leq \frac{\sqrt{1-x^2}}{2} \leq \frac{1}{2},$$

so $0 \leq z \leq 1/2$.

Next up, find bounds for x, note that because $x^2 + y^2 \leq 1$, and $y \geq 2z$,

$$|x| \leq \sqrt{1-y^2}$$
$$|x| \leq \sqrt{1-4z^2}$$

so $-\sqrt{1-4z^2} \leq x \leq \sqrt{1-4z^2}$.

Lastly, note that because $z \leq y/2$, y is bounded below as $2z \leq y$. So $2z \leq y \leq \sqrt{1-x^2}$. Then rewriting the integral, we have

$$\iiint_D f(x, y, z)\ dV = \int_0^{1/2} \int_{-\sqrt{1-4z^2}}^{\sqrt{1-4z^2}} \int_{2z}^{\sqrt{1-x^2}} f(x, y, z)\ dy\ dx\ dz. \qquad \blacksquare$$