MA2104 Assignment 4

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Question 1

Find

$$\iiint_E z(x^2+y^2+z^2)^{-3/2}\,dV$$

where

$$E := \{\,(x,y,z) : x^2+y^2+z^2 \le 16, z \ge 2\,\}\,.$$

Upper bound for ϕ happens at $z=2$, $x^2+y^2=12$, then $\rho=4$ and $\phi=\arccos(2/4)=\pi/3$.

Since $z=\rho\cos\phi$ and $2\le z$, we have $2\sec\phi\le\rho$.

Hence, converting to spherical coordinates, we have

$$E = \left\{\,(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\theta) : 0\le\theta\le 2\pi, 0\le\phi\le\frac{\pi}{3}, 2\sec\phi\le\rho\le 4\right\}$$

then computing the integral, we have

$$\begin{aligned}
\iiint_E z(x^2+y^2+z^2)^{-3/2}\,dV &= \int_0^{2\pi}\int_0^{\pi/3}\int_{2\sec\phi}^4 \rho\cos\phi(\rho^2)^{-3/2}(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta \\
&= 2\pi\int_0^{\pi/3}\int_{2\sec\phi}^4 \cos\phi\sin\phi\,d\rho\,d\phi \\
&= 2\pi\int_0^{\pi/3}(4-2\sec\phi)\cos\phi\sin\phi\,d\phi \\
&= 4\pi\int_0^{\pi/3} 2\sin\phi\cos\phi - \sin\phi\,d\phi \\
&= 4\pi\int_0^{\pi/3}\sin(2\phi)-\sin\phi\,d\phi \\
&= 4\pi\left[-\frac{\cos(2\phi)}{2}+\cos\phi\right]_0^{\pi/3} \\
&= 4\pi\left(\frac{1}{4}+\frac{1}{2}+\frac{1}{2}-1\right) = \pi
\end{aligned}$$

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Question 2

Find

$$\iint_D x\,dA$$

where

$$D := \{\,(x,y) : x^2+(y-1)^2 \le 1, x^2+y^2 \ge 1\,\}.$$

To convert polar coordinates, we need the ranges of (r,θ). To find range of θ, we first need to find points where the two circles intersect. By inspection, we have $y=\frac{1}{2}$. Solving $x^2+\frac{1}{4}=1$ gives the solution that $x=\pm\frac{\sqrt{3}}{2}$. Then

$$\arctan\left(\frac{1}{\sqrt{3}}\right) \le \theta \le \arctan\left(-\frac{1}{\sqrt{3}}\right)$$
$$\frac{\pi}{6} \le \theta \le \frac{5\pi}{6}$$

Next find bounds for r, it is obvious that $r \geq 1$, it remains to find upper bound.

$$\begin{aligned} x^2 + (y-1)^2 &\leq 1 \\ x^2 + y^2 - 2y &\leq 0 \\ r^2 &\leq 2r\sin\theta \\ r &\leq 2\sin\theta \end{aligned}$$

because $r \geq 1$. So

$$D = \left\{ (r\cos\theta, r\sin\theta) : \frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}, 1 \leq r \leq 2\sin\theta \right\}$$

Therefore,

$$\begin{aligned} \iint_D x\,dA &= \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} r\cos\theta\, r\,dr\,d\theta \\ &= \int_{\pi/6}^{5\pi/6} \cos\theta \left[\frac{r^3}{3}\right]_1^{2\sin\theta} dr \\ &= \int_{\pi/6}^{5\pi/6} \frac{8}{3}\sin^3\theta\cos\theta - \frac{1}{3}\cos\theta\,d\theta \\ &= \left[\frac{2}{3}\sin^4\theta - \frac{1}{3}\sin\theta\right]_{\pi/6}^{5\pi/6} \\ &= 0 \quad \because \sin\left(\frac{\pi}{6}\right) = \sin\left(\frac{5\pi}{6}\right) \end{aligned}$$

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Question 3

Find

$$\iint_D \frac{2x^2+y^2}{xy} dA$$

where

$$D := \{(x,y) : 1 \leq \frac{y}{\sqrt{x}} \leq 2, 1 \leq x^2 + y^2 \leq 4\}.$$

Let $u = x^2 + y^2$ and $v = \frac{y}{\sqrt{x}}$. Clearly we have $1 \leq u \leq 4$ and $1 \leq v \leq 2$. We have

$$\begin{aligned} \frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix} 2x & 2y \\ -\frac{1}{2}yx^{-3/2} & x^{-1/2} \end{vmatrix} \\ &= 2x^{1/2} + x^{-3/2}y^2 \end{aligned}$$

Using the identity that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\dfrac{\partial(x,y)}{\partial(u,v)}}$$

we can compute the change of variable

$$\begin{aligned}
\iint_D \frac{2x^2+y^2}{xy}\,dA &= \int_1^2 \int_1^4 \frac{2x^2+y^2}{xy} \cdot \frac{1}{2x^{1/2}+x^{-3/2}y^2}\,du\,dv \\
&= \int_1^2 \int_1^4 \frac{2x^2+y^2}{y\left(2x^{3/2}+x^{-1/2}y^2\right)}\,du\,dv \\
&= \int_1^2 \int_1^4 \frac{2x^2+y^2}{\dfrac{y}{\sqrt{x}}\left(2x^2+y^2\right)}\,du\,dv \\
&= \int_1^2 \int_1^4 \frac{1}{v}\,du\,dv \\
&= 3\left[\ln v\right]_1^2 = 3\ln 2
\end{aligned}$$

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