Models of Set Theory without Choice

Midterm Progress Talk

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Slides

https://m5th.b0ss.net/MA4199/slides-midterm.pdf

Forcing

Forcing Crash Course

- Start with M a transitive model of ZFC, consider $(\mathbb{P}, \leq, 1_{\mathbb{P}}) \in M$.
- $\mathbb{P}$-names $M^{\mathbb{P}} = \{(x, p) : x \in M^{\mathbb{P}}, p \in \mathbb{P}\}$ defined recursively.
- G generic over M when G meets every dense set in $\mathbb{P}$.

  Intuitively G encodes an object added to M which does not have any property that is definable in M.

Symmetric extension

- Consider $\mathcal{G}$ be a group of automorphisms $\mathbb{P} \to \mathbb{P}$,
- Each $\pi \in \mathcal{G}$ can be naturally extended into an automorphism $\pi_* : M^{\mathbb{P}} \to M^{\mathbb{P}}$ by

$$\pi_*(x, p) = \{(\pi_* y, \pi p) : y \in x\}.$$

- Given a normal filter (on algebra of subgroups of $\mathcal{G}$) $\mathcal{F}$ we can use it to define the notion of *symmetric*. A $x \in M^{\mathbb{P}}$ is symmetric if its stabilizer is in the filter.

  Normal filters are also closed under conjugation, if $\pi \in \mathcal{G}$ and $H \in \mathcal{F}$ then $\pi H \pi^{-1} \in \mathcal{F}$.

Symmetric extension

- $x \in M^{\mathbb{P}}$ is a hereditarily symmetric name if x is symmetric and every member in $\operatorname{dom}(x)$ is hereditarily symmetric. Denote them **HS**.
- Consider subset of $M[G]$ formed by using G to evaluate all the names in **HS**
$$N = \{x[G] : x \in M^{\mathbb{P}}\}.$$
- $M \subseteq N \subseteq M[G]$

  Check: $N \vDash$ **ZF**.

Cohen models

Basic Cohen model

- The first model produced in which choice fails (showing independence).
- Force with $\mathbb{P}$ finite partial functions $\omega \times \omega \rightharpoonup 2$, ordered by reverse containment $\leq = \supseteq$.
- Idea: add ω many reals satisfying

$$x_n = \{m \in \omega : (\exists p \in G)\, p(n,m) = 1\}\,.$$

  N should contain $A = \{x_n : n \in \omega\}$ but not know how to well-order it.
- Come up with names for objects we wish to add

$$\mathring{x}_n = \{(\check{m}, p) : m \in \omega, p \in \mathbb{P}, p(n,m) = 1\}$$

$$\mathring{A} = \{(\mathring{x}_n, 1_{\mathbb{P}}) : n \in \omega\}$$

Basic Cohen model

- Any permutation $\pi_0 : \omega \to \omega$ induces an automorphism $\pi : \mathbb{P} \to \mathbb{P}$ as

$$\pi p = \{((\pi_0 n, m), y) : ((n, m), y) \in p\}.$$

- Let $\mathcal{G}$ be all such induced permutations on $\mathbb{P}$, then for any finite $B \subset \omega$, we look at permutations induced by ones in its stabilizer

$$\mathrm{fix}(B) = \{\pi \in \mathcal{G} : \forall n \in B\, (\pi_0 n = n)\}.$$

- Use stabilizers of all finite subsets to generate a (normal) filter

$$\mathcal{F} = \{H : \exists \text{ finite } B \subseteq \omega\, (\mathrm{fix}(B) \subseteq H)\}.$$

- Let $\mathcal{G}$ and $\mathcal{F}$ determine **HS** and if we let G be a generic filter, we yield a symmetric model N.

Failure of choice in basic Cohen model

- Check: $\pi_* \mathring{x}_n = \mathring{x}_{\pi n}$

  $\mathring{x}_n$ is symmetric as its stabilizer contains $\mathrm{fix}(\{n\})$.

- Claim: $N \vDash x_i \neq x_j$ whenever $i \neq j$

  Suppose $p \in \mathbb{P}$ such that $p \Vdash \mathring{x}_i = \mathring{x}_j$. p is finite so we can find an extension $q \leq p$ that forces $m \in x_i \wedge m \notin x_j$ for some $m \in \omega$.

Claim: **There does not exist a bijection $f : \omega \to A$.**

- Suppose there is, let $p_0 \in G$ such that

$$p_0 \Vdash \mathring{f} \text{ is a bijection } \check{\omega} \to \mathring{A}.$$

Failure of choice in basic Cohen model

- $f \in N$ so $\mathring{f}$ is symmetric. Let $B \subseteq \omega$ be finite such that $\operatorname{fix}(B)$ is contained in stabilizer of $\mathring{f}$.
- Let $n \notin B, i \in \omega, p \leq p_0$ such that

$$p \Vdash \mathring{f}(\check{i}) = \mathring{x}_n.$$

- Now we can find $\pi \in \mathcal{G}$ satisfying

  i. πp compatible with p
  ii. $\pi \in \operatorname{fix}(B)$
  iii. $\pi n \neq n$

- Then $\pi p \Vdash (\pi \mathring{f})(\pi \check{i}) = \pi \mathring{x}_n$, so

$$p \cup \pi p \Vdash \mathring{f}(\check{i}) = \mathring{x}_n \wedge \mathring{f}(\check{i}) = \mathring{x}_{\pi n}.$$

Second Cohen model

- Countable choice fails (N knows that our misbehaving set is countable).
- Force with $\mathbb{P}$ finite partial functions $(\omega \times 2 \times \omega) \times \omega \rightharpoonup 2$, ordered by reverse containment $\leq = \supseteq$.

Define the notions ($\mathbb{P}$-names omitted)

- $x_{n\varepsilon i} = \{j \in \omega : (\exists p \in G)\,(p(n\varepsilon i, j) = 1)\}$
- $X_{n\varepsilon} = \{x_{n\varepsilon i} : i \in \omega\}$
- $P_n = \{X_{n0}, X_{n1}\}$
- $A = \{P_n : n \in \omega\}$

Second Cohen model

Again we can extend permutations on $(\omega \times 2 \times \omega)$ to permutations on $\mathbb{P}$ naturally

$$\pi p = \{((\pi_0(n\varepsilon i), j), y) : (((n\varepsilon i), j), y) \in p\}.$$

Restrict our attention to permutations π_0 satisfying

i. $n' = n$
ii. for each n either $\forall i\,(\varepsilon' = \varepsilon)$ or $\forall i\,(\varepsilon' \neq \varepsilon)$

Consider the automorphism group $\mathcal{G}$ of all such permutations extended to $\mathbb{P}$ then to $M^{\mathbb{P}}$, for any finite $B \subset (\omega \times 2 \times \omega)$,

$$\operatorname{fix}(B) = \{\pi \in \mathcal{G} : \forall n \in B\,(\pi_0 n = n)\}.$$

Again we can generate a normal filter, and proceed to get symmetric model.

Second Cohen model

- Check: $x_{n\varepsilon i}, X_{n\varepsilon}, P_n, A$ all in N (their names are symmetric).
- Claim: $N \models A$ is countable.

  Define $\mathring{g} = \left\{((\check{n}, \mathring{P}_n), 1_{\mathbb{P}}) : n \in \omega\right\}$, check that $\mathring{g}$ is **HS**.
  Evaluating $\mathring{g}$ will enumerate $\langle P_n : n \in \omega\rangle$.

Failure of choice in second Cohen model

Suppose f is a choice function on A, let $\mathring{f} \in$ **HS** be its symmetric name and $p_0 \in G$ force

$$p_0 \Vdash \mathring{f} \text{ is a function with domain } \mathring{A} \text{ and } \mathring{f}(\mathring{P}_n) \in \mathring{P}_n \text{ for all } n.$$

Let $B \subseteq \omega \times 2 \times \omega$ such that stabiliser of $\mathring{f}$ contains $\mathrm{fix}(B)$, let $n \in \omega, p \leq p_0, \varepsilon_0$ such that (wlog suppose $\varepsilon_0 = 0$)

$$p \Vdash \mathring{f}(\mathring{P}_n) = \mathring{X}_{n0}.$$

We can find $\pi \in \mathcal{G}$ satisfying

i. πp compatible with p
ii. $\pi \in \mathrm{fix}(B)$
iii. $\pi(\mathring{X}_{n0}) = \pi(\mathring{X}_{n1})$

Failure of choice in second Cohen model

How? Observe that i-coordinate is "free", we are free to shuffle the sequence $\langle x_{n\varepsilon i} : i \in \omega \rangle$.

Let π be extended from

$$\pi_0(n,0,i) = \begin{cases} (n,1,i+k) & \text{when} i < k \\ (n,1,i-k) & \text{when} k \le i < 2k \\ (n,1,i) & \text{otherwise} \end{cases}$$

$$\pi_0(n,1,i) = \text{similar to above}$$

and identity everywhere else.

With π obtained, use similar forcing argument as Cohen basic model to get

$$N \vDash f \text{ is not a function.}$$

Next steps

- Every set of real is Lesbegue measurable (Solovay).
- All uncountable cardinals being singular (Gitik).

Thank you