Week 10 Exercises

Qi Ji

1st April 2020

When we define $\neg A := A \to \bot$, it is clearer to discard the $\neg\mathrm{L}$ and $\neg\mathrm{R}$ rules and only use the arrow ones. It also turns out that I typed the majority of stuff below while using a fake "$\to \mathrm{L}$" that is more symmetric to $\to \mathrm{R}$, which is

$$\frac{\Gamma \Rightarrow A \to B}{\Gamma, A \Rightarrow B}\ \text{“}\!\to \mathrm{L}\text{”}$$

so let's just show that "→L" rule is admissible using the actual →L rule.

$$\text{cut}\ \dfrac{\begin{array}{c}\vdots \\ \Gamma \Rightarrow A \to B\end{array} \qquad \dfrac{A \Rightarrow A \qquad B \Rightarrow B}{A \to B, A \Rightarrow B}\ {\to}\mathrm{L}}{\Gamma, A \Rightarrow B}$$

I don't think we can derive my fake rule using the real one, since

$$\text{cut}\ \dfrac{\text{cut}\ \dfrac{\begin{array}{c}\vdots \\ \Gamma \Rightarrow A\end{array} \qquad \text{“}\!{\to}\mathrm{L}\text{”}\ \dfrac{\begin{array}{c}\vdots\ \text{need real}\ {\to}\mathrm{L}\ \text{here!} \\ A \to B \Rightarrow A \to B\end{array}}{A \to B, A \Rightarrow B}}{\Gamma, A \to B \Rightarrow B} \qquad \begin{array}{c}\vdots \\ B, \Gamma \Rightarrow \Delta\end{array}}{\Gamma, A \to B \Rightarrow \Delta}$$

the axiom pseudo-rule for non-atomic formulas is derived using the real →L rule.

In any case for everything below, →L refers to the fake one unless otherwise stated.

Lemma 1

> $I \vdash \neg\neg A^N \Rightarrow A^N$.

Case A atomic, then we have a special case of $I \vdash \neg\neg\neg\tau \Rightarrow \neg\tau$

$$
\begin{prooftree}
\AxiomC{\(\neg\tau \Rightarrow \neg\tau\)}
\RightLabel{$\to$L}
\UnaryInfC{\(\tau, \neg\tau \Rightarrow \bot\)}
\RightLabel{$\to$R}
\UnaryInfC{\(\tau \Rightarrow \neg\neg\tau\)}
\AxiomC{\(\neg\neg\neg\tau \Rightarrow \neg\neg\tau \to \bot\)}
\RightLabel{$\to$L}
\UnaryInfC{\(\neg\neg\neg\tau, \neg\neg\tau \Rightarrow \bot\)}
\RightLabel{cut}
\BinaryInfC{\(\tau, \neg\neg\neg\tau \Rightarrow \bot\)}
\RightLabel{$\to$R}
\UnaryInfC{\(\neg\neg\neg\tau \Rightarrow \neg\tau\)}
\end{prooftree}
$$

Case A is $B \wedge C$, then assume we already have $I \vdash \neg\neg B^N \Rightarrow B^N$ and $I \vdash \neg\neg C^N \Rightarrow C^N$, we want $I \vdash \neg\neg(B^N \wedge C^N) \Rightarrow B^N \wedge C^N$.

$$
\begin{prooftree}
\AxiomC{\(B^N \Rightarrow B^N\)}
\RightLabel{$\wedge$L}
\UnaryInfC{\(B^N \wedge C^N \Rightarrow B^N\)}
\dashedLine
\RightLabel{cut with $B^N, \neg B^N \Rightarrow \bot$}
\UnaryInfC{\(\neg B^N, B^N \wedge C^N \Rightarrow \bot\)}
\RightLabel{$\to$ R}
\UnaryInfC{\(\neg B^N \Rightarrow \neg(B^N \wedge C^N)\)}
\AxiomC{\(\vdots\, \tau, \neg\tau \Rightarrow \bot\)}
\noLine
\UnaryInfC{\(\neg(B^N \wedge C^N), \neg\neg(B^N \wedge C^N) \Rightarrow \bot\)}
\LeftLabel{cut}
\BinaryInfC{\(\neg B^N, \neg\neg(B^N \wedge C^N) \Rightarrow \bot\)}
\LeftLabel{$\to$R}
\UnaryInfC{\(\neg\neg(B^N \wedge C^N) \Rightarrow \neg\neg B^N\)}
\dashedLine
\LeftLabel{cut with IH}
\UnaryInfC{\(\neg\neg(B^N \wedge C^N) \Rightarrow B^N\)}
\end{prooftree}
$$

similar for C^N then $\wedge R$ gives $I \vdash \neg\neg(B^N \wedge C^N) \Rightarrow B^N \wedge C^N$ as desired.

Case A is $B \vee C$, assume the same IH and we want $I \vdash \neg\neg\neg(\neg B^N \wedge \neg C^N) \Rightarrow \neg(\neg B^N \wedge \neg C^N)$ but this is an instance of "triple negation entails single negation".

Case A is $B \to C$ is in lecture slides.

Case A is $\forall x B(x)$, assume $I \vdash \neg\neg B^N(t) \Rightarrow B^N(t)$ with occurences of t suitably indicated and we want $I \vdash \neg\neg\forall x B^N(x) \Rightarrow \forall x B^N(x)$.

$$
\begin{prooftree}
\AxiomC{\(B^N(t) \Rightarrow B^N(t)\)}
\LeftLabel{$\forall$L}
\UnaryInfC{\(\forall x B^N(x) \Rightarrow B^N(t)\)}
\AxiomC{\(\vdots\, \tau, \neg\tau \Rightarrow \bot\)}
\noLine
\UnaryInfC{\(B^N(t), \neg B^N(t) \Rightarrow \bot\)}
\LeftLabel{cut}
\BinaryInfC{\(\neg B^N(t), \forall x B^N(x) \Rightarrow \bot\)}
\LeftLabel{$\to$R}
\UnaryInfC{\(\neg B^N(t) \Rightarrow \neg\forall x B^N(x)\)}
\AxiomC{\(\vdots\, \tau, \neg\tau \Rightarrow \bot\)}
\noLine
\UnaryInfC{\(\neg\neg\forall x B^N(x), \neg\forall x B^N(x) \Rightarrow \bot\)}
\RightLabel{cut}
\BinaryInfC{\(\neg\neg\forall x B^N(x), \neg B^N(t) \Rightarrow \bot\)}
\RightLabel{$\to$R}
\UnaryInfC{\(\neg\neg\forall x B^N(x) \Rightarrow \neg\neg B^N(t)\)}
\dashedLine
\RightLabel{cut with IH}
\UnaryInfC{\(\neg\neg\forall x B^N(x) \Rightarrow B^N(t)\)}
\RightLabel{$\forall$R}
\UnaryInfC{\(\neg\neg\forall x B^N(x) \Rightarrow \forall x B^N(x)\)}
\end{prooftree}
$$

where in the $\forall$R application the eigenvariable condition is satisfied.

Case A is $\exists x B(x)$, apply "triple negation entails single negation".

Lemma 2

> If $\vdash \Gamma \Rightarrow \Delta$ then $I \vdash \Gamma^N, \neg\Delta^N \Rightarrow \bot$.

I just interpret $\Gamma \Rightarrow$ as $\Gamma \Rightarrow \bot$ and add the axiom $\bot \Rightarrow B$ for any formula B, which gives back *ex falso quodlibet*. The advantage of doing so is all intuitionistic sequents will have **exactly** 1 formula on the RHS instead of at most 1.

Cases where the last derivation is axiom and $\to$R are in slides.

Case where the last derivation is $\neg$L,

$$\dfrac{\Gamma_1 \Rightarrow \Delta, A}{\Gamma_1, \neg A \Rightarrow \Delta}\ \neg\text{L}$$

I'm interpreting $(\neg A)^N$ as $\neg A^N$. This is sensible as $(\neg A)^N$ expands to $(A \to \bot)^N$, and treating $\bot$ as atomic $(\neg A)^N = A^N \to \bot^N$ which is intuitionistically equivalent to $A^N \to \bot = \neg A^N$. So we want $I \vdash \Gamma_1^N, \neg A^N, \neg\Delta^N \Rightarrow \bot$ which is just the IH.

Case where the last derivation is $\neg$R,

$$\dfrac{\Gamma, A \Rightarrow \Delta_1}{\Gamma \Rightarrow \Delta_1, \neg A}\ \neg\text{R}$$

we want $I \vdash \Gamma^N, \neg\Delta_1^N, \neg\neg A^N \Rightarrow \bot$,

$$\dfrac{\to\text{R}\ \dfrac{\dfrac{\vdots\ \text{IH}}{\Gamma^N, A^N, \neg\Delta_1^N \Rightarrow \bot}}{\Gamma^N, \neg\Delta_1^N \Rightarrow \neg A^N} \qquad \dfrac{\vdots\ \tau, \neg\tau \Rightarrow \bot}{\neg A^N, \neg\neg A^N \Rightarrow \bot}}{\Gamma^N, \neg\Delta_1^N, \neg\neg A^N \Rightarrow \bot}\ \text{cut}$$

Case where the last derivation is $\to$L (the classical one). Then for some $A \to B$ we have

$$\dfrac{\Gamma_1 \Rightarrow \Delta_1, A \qquad B, \Gamma_1 \Rightarrow \Delta_2}{A \to B, \Gamma_1 \Rightarrow \Delta_1, \Delta_2}\ \to\text{L}$$

we want to show $I \vdash A^N \to B^N, \Gamma_1^N, \neg\Delta_1^N, \neg\Delta_2^N \Rightarrow \bot$,

$$
\begin{prooftree}
\AxiomC{$\vdots$ IH}
\noLine
\UnaryInfC{$\Gamma_1^N, \neg\Delta_1^N, \neg A^N \Rightarrow \bot$}
\LeftLabel{$\to$R}
\UnaryInfC{$\Gamma_1^N, \neg\Delta_1^N \Rightarrow \neg\neg A^N$}
\LeftLabel{cut with Lemma 1}
\dashedLine
\UnaryInfC{$\Gamma_1^N, \neg\Delta_1^N \Rightarrow A^N$}
\AxiomC{$\vdots$ IH}
\noLine
\UnaryInfC{$\Gamma_1^N, \neg\Delta_2^N, B^N \Rightarrow \bot$}
\LeftLabel{weaken, ACTUAL (not fake) $\to$L}
\BinaryInfC{$\Gamma_1^N, \neg\Delta_1^N, \neg\Delta_2^N, A^N \to B^N \Rightarrow \bot$}
\end{prooftree}
$$

Case last derivation is $\wedge$L1

$$
\begin{prooftree}
\AxiomC{$\Gamma_1, B \Rightarrow \Delta$}
\RightLabel{$\wedge$L1}
\UnaryInfC{$\Gamma_1, B \wedge C \Rightarrow \Delta$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\vdots$ IH}
\noLine
\UnaryInfC{$\Gamma_1^N, \neg\Delta^N, B^N \Rightarrow \bot$}
\LeftLabel{$\to$R}
\UnaryInfC{$\Gamma_1^N, \neg\Delta^N, \Rightarrow \neg B^N$}
\AxiomC{$\vdots$Lem 1 case $\wedge$}
\noLine
\UnaryInfC{$\neg B^N \Rightarrow \neg(B^N \wedge C^N)$}
\RightLabel{cut}
\BinaryInfC{$\Gamma_1^N, \neg\Delta^N \Rightarrow B^N \wedge C^N \to \bot$}
\RightLabel{$\to$L}
\UnaryInfC{$\Gamma_1^N, \neg\Delta^N, B^N \wedge C^N \Rightarrow \bot$}
\end{prooftree}
$$

and similarly for $\wedge$L2.

Suppose last derivation is $\wedge$R,

$$
\begin{prooftree}
\AxiomC{$\Gamma \Rightarrow \Delta_1, A$}
\AxiomC{$\Gamma \Rightarrow \Delta_1, B$}
\RightLabel{$\wedge$R}
\BinaryInfC{$\Gamma \Rightarrow \Delta_1, A \wedge B$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\vdots$ IH}
\noLine
\UnaryInfC{$\Gamma^N, \neg\Delta_1^N, \neg A^N \Rightarrow \bot$}
\LeftLabel{$\to$R}
\UnaryInfC{$\Gamma^N, \neg\Delta_1^N \Rightarrow \neg\neg A^N$}
\LeftLabel{cut with lemma 1}
\dashedLine
\UnaryInfC{$\Gamma^N, \neg\Delta_1^N \Rightarrow A^N$}
\AxiomC{$\vdots$ with the other IH}
\noLine
\UnaryInfC{$\Gamma^N, \neg\Delta_1^N \Rightarrow B^N$}
\RightLabel{$\wedge$R}
\BinaryInfC{$\Gamma^N, \neg\Delta_1^N \Rightarrow A^N \wedge B^N$}
\RightLabel{cut with $\tau \Rightarrow \neg\neg\tau$}
\dashedLine
\UnaryInfC{$\Gamma^N, \neg\Delta_1^N \Rightarrow \neg\neg(A^N \wedge B^N)$}
\RightLabel{$\to$L}
\UnaryInfC{$\Gamma^N, \neg\Delta_1^N, \neg(A^N \wedge B^N) \Rightarrow \bot$}
\end{prooftree}
$$

as desired, since $(A \wedge B)^N = A^N \wedge B^N$.

Suppose last derivation is $\vee$L,

$$\dfrac{A, \Gamma_1 \Rightarrow \Delta \qquad B, \Gamma_1 \Rightarrow \Delta}{A \lor B, \Gamma_1 \Rightarrow \Delta} \lor\mathrm{L}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathrm{IH}\\ \Gamma_1^N, \neg\Delta^N, A^N \Rightarrow \bot\end{array}}{\Gamma_1^N, \neg\Delta^N \Rightarrow \neg A^N}{\to}\mathrm{R} \qquad \begin{array}{c}\vdots\,\text{the other IH}\\ \Gamma_1^N, \neg\Delta^N \Rightarrow \neg B^N\end{array}}{\Gamma_1^N, \neg\Delta^N \Rightarrow \neg A^N \land \neg B^N} \land\mathrm{R}}{\Gamma_1^N, \neg\Delta^N \Rightarrow \neg\neg(\neg A^N \land \neg B^N)} \text{ cut with } \tau \Rightarrow \neg\neg\tau}{\Gamma_1^N, \neg\Delta^N, \neg(\neg A^N \land \neg B^N) \Rightarrow \bot} {\to}\mathrm{L}$$

as desired, note $(A \lor B)^N = \neg(\neg A^N \land \neg B^N)$.

Suppose last derivation is $\lor$R1,

$$\dfrac{\Gamma \Rightarrow \Delta_1, A}{\Gamma \Rightarrow \Delta_1, A \lor B} \lor\mathrm{R1}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\mathrm{IH}\\ \Gamma^N, \neg\Delta_1^N, \neg A^N \Rightarrow \bot\end{array}}{\Gamma^N, \neg\Delta_1^N \Rightarrow \neg\neg A^N}{\to}\mathrm{R} \qquad \begin{array}{c}\vdots\,\text{similar to Lem 1 case } \land\\ \neg(\neg A^N) \Rightarrow \neg(\neg A^N \land \neg B^N)\end{array}}{\Gamma^N, \neg\Delta_1^N \Rightarrow \neg(\neg A^N \land \neg B^N)} \mathrm{cut}}{\Gamma^N, \neg\Delta_1^N \Rightarrow \neg\neg\neg(\neg A^N \land \neg B^N)} \text{ cut with } \tau \Rightarrow \neg\neg\tau}{\Gamma^N, \neg\Delta_1^N, \neg(\neg A^N \land \neg B^N) \Rightarrow \bot} {\to}\mathrm{L}$$

as desired. Similar for $\lor$R2.

Suppose last derivation is $\forall$L,

$$\dfrac{F(t), \Gamma_1 \Rightarrow \Delta}{\forall x F(x), \Gamma_1 \Rightarrow \Delta} \forall\mathrm{L}$$

$$\dfrac{\begin{array}{c}\vdots\,\mathrm{IH}\\ \Gamma_1^N, \neg\Delta^N, F^N(t) \Rightarrow \bot\end{array}}{\Gamma_1^N, \neg\Delta^N, \forall x F^N(x) \Rightarrow \bot} \forall\mathrm{L}$$

Suppose last derivation is $\exists$R,

$$\dfrac{\Gamma \Rightarrow \Delta_1, F(t)}{\Gamma \Rightarrow \Delta_1, \exists x F(x)} \exists\mathrm{R}$$

$$
\begin{prooftree}
\AxiomC{$\vdots$ IH}
\noLine
\UnaryInfC{$\Gamma^N, \neg\Delta_1^N, \neg F^N(t) \Rightarrow \bot$}
\RightLabel{$\forall$L}
\UnaryInfC{$\Gamma^N, \neg\Delta_1^N, \forall x \neg F^N(x) \Rightarrow \bot$}
\RightLabel{$\to$R}
\UnaryInfC{$\Gamma^N, \neg\Delta_1^N \Rightarrow \neg\forall x \neg F^N(x)$}
\dashedLine
\RightLabel{cut with $\tau \Rightarrow \neg\neg\tau$}
\UnaryInfC{$\Gamma^N, \neg\Delta_1^N \Rightarrow \neg\neg\neg\forall x \neg F^N(x)$}
\RightLabel{$\to$L}
\UnaryInfC{$\Gamma^N, \neg\Delta_1^N, \neg\neg\forall x \neg F^N(x) \Rightarrow \bot$}
\end{prooftree}
$$

note $\neg\forall\neg F^N(x) = (\exists x F(x))^N$ as desired.

Suppose last derivation is $\forall$R with eigenvariable a,

$$
\begin{prooftree}
\AxiomC{$\Gamma \Rightarrow \Delta_1, F(a)$}
\RightLabel{$\forall$R}
\UnaryInfC{$\Gamma \Rightarrow \Delta_1, \forall x F(x)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\vdots$ IH}
\noLine
\UnaryInfC{$\Gamma^N, \neg\Delta_1^N, \neg F^N(a) \Rightarrow \bot$}
\RightLabel{$\to$R}
\UnaryInfC{$\Gamma^N, \neg\Delta_1^N \Rightarrow \neg\neg F^N(a)$}
\dashedLine
\LeftLabel{cut with Lemma 1}
\UnaryInfC{$\Gamma^N, \neg\Delta_1^N \Rightarrow F^N(a)$}
\RightLabel{$\forall$R}
\UnaryInfC{$\Gamma^N, \neg\Delta_1^N \Rightarrow \forall x F^N(x)$}
\dashedLine
\LeftLabel{cut with $\tau \Rightarrow \neg\neg\tau$}
\UnaryInfC{$\Gamma^N, \neg\Delta_1^N \Rightarrow \neg\neg\forall x F^N(x)$}
\UnaryInfC{$\Gamma^N, \neg\Delta_1^N, \neg\forall x F^N(x) \Rightarrow \bot$}
\end{prooftree}
$$

Suppose last derivation is $\exists$L with eigenvariable a,

$$
\begin{prooftree}
\AxiomC{$F(a), \Gamma_1 \Rightarrow \Delta$}
\RightLabel{$\exists$L}
\UnaryInfC{$\exists x F(x), \Gamma_1 \Rightarrow \Delta$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\vdots$ IH}
\noLine
\UnaryInfC{$\Gamma_1^N, \neg\Delta^N, F^N(a) \Rightarrow \bot$}
\RightLabel{$\to$R}
\UnaryInfC{$\Gamma_1^N, \neg\Delta^N \Rightarrow \neg F^N(a)$}
\RightLabel{$\forall$R}
\UnaryInfC{$\Gamma_1^N, \neg\Delta^N \Rightarrow \forall x \neg F^N(x)$}
\dashedLine
\LeftLabel{cut with $\tau \Rightarrow \neg\neg\tau$}
\UnaryInfC{$\Gamma_1^N, \neg\Delta^N \Rightarrow \neg\neg\forall x \neg F^N(x)$}
\UnaryInfC{$\Gamma_1^N, \neg\Delta^N, \neg\forall x \neg F^N(x) \Rightarrow \bot$}
\end{prooftree}
$$

as desired since $(\exists x F(x))^N = \neg\forall x \neg F^N(x)$.