MA6222 Exercise for Week 5

Qi Ji

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1

> Prove $\alpha \mapsto \varphi_\alpha(0)$ is continuous.

Let $f(\alpha) = \varphi_\alpha(0)$ and $U \subseteq \mathbf{ON}$ be a nonempty set, we want $f(\sup U) = \sup f[U]$.

In the case where $\sup U = \max U$ result easily follows as f preserves order, so without loss of generality we consider the case where $\lambda = \sup U$ is a limit.

It suffices to show that

$$\sup f[U] = \sup_{\xi \in U} \min \mathrm{Cr}(\xi) \stackrel{?}{=} \min \bigcap_{\xi \in U} \mathrm{Cr}(\xi) = \min \bigcap_{\xi < \lambda} \mathrm{Cr}(\xi) = f(\lambda).$$

For any $\zeta \in U$,

$$\min \mathrm{Cr}(\zeta) \leq \min \bigcap_{\xi \in U} \mathrm{Cr}(\xi).$$

Conversely we check that for all $\zeta \in U$,

$$\sup_{\xi \in U} \min \mathrm{Cr}(\xi) = \sup_{\xi \in U, \xi > \zeta} \min \mathrm{Cr}(\xi) \in \mathrm{Cr}(\zeta)$$

as $\xi > \zeta$ implies $\mathrm{Cr}(\xi) \subsetneq \mathrm{Cr}(\zeta)$ and $\mathrm{Cr}(\zeta)$ is closed, therefore $\sup_{\xi \in U} \min \mathrm{Cr}(\xi) \in \bigcap_{\xi \in U} \mathrm{Cr}(\xi)$. Now if $\alpha < \sup_{\xi \in U} \min \mathrm{Cr}(\xi)$, then $\alpha < \min \mathrm{Cr}(\zeta)$ for some $\zeta \in U$ which means $\alpha \notin \bigcap_{\xi \in U} \mathrm{Cr}(\xi)$. This shows the minimality.

2

> $\mathrm{SC} := \{\alpha : \varphi_\alpha(0) = \alpha\}$. Prove lemma 4.30 which says $\mathrm{SC} = \{\alpha : \alpha > 0 \wedge \forall \xi, \eta < \alpha, \varphi_\xi(\eta) < \alpha\}$.

In general, each $\beta \in \mathrm{Cr}(\alpha)$, satisfies for all $\xi < \alpha$, $\varphi_\xi[\beta] \subseteq \beta$. To see this let $\beta = \varphi_\alpha(\eta)$ and let $\gamma < \beta$, then by Theorem 4.25 (ii) 1

$$\varphi_\xi(\gamma) < \beta = \varphi_\alpha(\eta).$$

Now when $\alpha \in \mathrm{SC}$, then as $\alpha \in \mathrm{Cr}(\alpha)$ the desired closure property follows.

Conversely suppose $\alpha > 0$ satisfies the closure property, to show $\alpha \in \mathrm{SC}$ it suffices to show that $\varphi_\alpha(0) \leq \alpha$. We first immediately observe that for all $\xi, \eta < \alpha$, $\xi + \eta \leq \varphi_\xi(0) + \eta \leq \varphi_\xi(\eta) < \alpha$, so $\alpha \in \mathbb{AP}$.

By induction on β we show $\beta < \varphi_\alpha(0) \implies \beta < \alpha$. If $\beta \notin \mathbb{AP}$ just use induction hypothesis as $\alpha \in \mathbb{AP}$. Now for the case that $\beta \in \mathbb{AP}$, $\beta = \varphi_\xi(\eta)$ for $\eta < \beta$. By φ-comparison, the only valid case for

$$\varphi_\xi(\eta) < \varphi_\alpha(0)$$

entails that $\xi < \alpha$, and by induction hypothesis $\eta < \alpha$ too, so $\beta = \varphi_\xi(\eta) < \alpha$.

3

Define $\psi : \varepsilon_0 \to \mathcal{F}$ recursively as follows,

$$\psi(0)(x) = 0$$

and whenever $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$,

$$\psi(\alpha)(x) = x^{\psi(\alpha_1)(x)} + \cdots + x^{\psi(\alpha_n)(x)}$$

we can structurally induct to check that f is onto.

It remains to show $\alpha < \beta \implies \psi(\alpha) \prec \psi(\beta)$. We induct on β. Suppose $0 < \alpha < \beta$ ($\alpha = 0$ case is trivial) and

$$\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_m}$$
$$\beta =_{NF} \omega^{\beta_1} + \cdots + \omega^{\beta_n}$$

now compare the normal forms.

Case $m < n$ and $\alpha_i = \beta_i$ for all $i \leq m$, then

$$\psi(\alpha) + \text{some nonzero function} = \psi(\beta)$$

so $\psi(\alpha) \prec \psi(\beta)$.

Case there exists j such that $\alpha_j < \beta_j$ and $\alpha_i = \beta_i$ for all $1 \leq i < j$. By induction hypothesis as $\psi(\alpha_j) \prec \psi(\beta_j)$, so let k_0 such that for all $x \geq k_0$

$$\begin{gathered}
\psi(\alpha_j)(x) + 1 \leq \psi(\beta_j)(x) \\
x \cdot x^{\psi(\alpha_j)(x)} \leq x^{\psi(\beta_j)(x)} \\
\underbrace{x^{\psi(\alpha_j)(x)} + \cdots + x^{\psi(\alpha_j)(x)}}_{x \text{ copies}} \leq x^{\psi(\beta_j)(x)}
\end{gathered}$$

Now choose $k \geq \max(m - j + 1, k_0)$ that also witnesses $\psi(\alpha_i) \prec \psi(\alpha_j)$ for all $j < i \leq m$, that is whenever $x \geq k$, $\psi(\alpha_i)(x) < \psi(\alpha_j)(x)$.

Whenever $x \geq k$,

$$\begin{aligned}
x^{\psi(\alpha_j)(x)} + \cdots + x^{\psi(\alpha_m)(x)} &\leq \underbrace{x^{\psi(\alpha_j)(x)} + \cdots + x^{\psi(\alpha_j)(x)}}_{m-j \text{ copies}} \\
&< \underbrace{x^{\psi(\alpha_j)(x)} + \cdots + x^{\psi(\alpha_j)(x)}}_{x \text{ copies}} \\
&\leq x^{\psi(\beta_j)(x)} \\
&\leq x^{\psi(\beta_j)(x)} + \cdots + x^{\psi(\beta_n)(x)}
\end{aligned}$$

so it follows that

$$\psi(\alpha) \prec \psi(\beta).$$

So by induction ψ witnesses the isomorphism $(\varepsilon_0, <) \cong (\mathcal{F}, \prec)$.

The order type of one variable polynomials under $\prec$ is $\omega^\omega = \psi^{-1}\{\text{one variable polynomials}\}$.